

A POSITIVITY CONJECTURE FOR THE ALVIS-CURTIS DUAL OF THE INTERSECTION COHOMOLOGY OF A DELIGNE–LUSZTIG VARIETY

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ABSTRACT. We formulate a strong positivity conjecture on characters afforded by the Alvis-Curtis dual of the intersection cohomology of Deligne–Lusztig varieties. This conjecture provides a powerful tool to determine decomposition numbers of unipotent ℓ -blocks of finite reductive groups.

1. INTRODUCTION

An important part of the modular representation theory of a finite group is encoded in its decomposition matrix, which relates simple representations in characteristic zero with those in positive characteristic. Determining such matrices amounts to finding the characters of the indecomposable projective representations.

In the case of finite reductive groups, projective modules can be constructed by Harish-Chandra induction. However, like in the ordinary case, many modules do not appear in the induction from proper Levi subgroups. They correspond to the projective covers of the so-called cuspidal modules. For these, one needs to use Deligne–Lusztig induction instead of Harish-Chandra induction, the problem being that the latter produces virtual modules. We conjecture that one can obtain proper projective modules, and so overcome this problem, by considering suitable linear combinations of these virtual modules.

Let us give more details about the construction. Let \mathbf{G} be a connected reductive algebraic group over an algebraically closed field of positive characteristic with a Steinberg endomorphism F making \mathbf{G}^F into a finite reductive group. To any element w of the Weyl group of \mathbf{G} one can associate a quasi-projective variety $X(w)$ acted on by \mathbf{G}^F , the Deligne–Lusztig variety. The alternating sum of the ℓ -adic cohomology groups of $X(w)$ yields a virtual character R_w of \mathbf{G}^F . Taking instead the Alvis-Curtis dual of the intersection cohomology one obtains another virtual character Q_w , which is a linear combination of the Deligne–Lusztig characters R_y . The coefficients of this combination are the values at 1 of the twisted Kazhdan–Lusztig polynomials, which also express simple objects in terms of Verma modules in the principal block of the category \mathcal{O} of a semi-simple Lie algebra. Surprisingly, unlike the Deligne–Lusztig characters the Q_w ’s are no longer virtual. The purpose of this note is to conjecture a modular analogue of that property and to give evidence towards it (see §2.3).

Conjecture 1.1. *Assume that ℓ is not too small. Then up to a sign, Q_w is the unipotent part of a projective character.*

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We shall explain the strong implications of this conjecture to the determination of decomposition matrices for finite reductive groups, and give several instructive examples where this conjecture holds. We believe that a general proof might rely on the geometric realization of the Alvis-Curtis duality.

2. STATEMENT OF THE CONJECTURE

2.1. Deligne–Lusztig theory. Let \mathbf{G} be a connected reductive linear algebraic group over an algebraically closed field of positive characteristic p , and $F : \mathbf{G} \rightarrow \mathbf{G}$ be a Steinberg endomorphism. There exists a positive integer δ such that F^δ defines a split \mathbb{F}_{q^δ} -structure on \mathbf{G} (with $q \in \mathbb{R}_+$), and we will choose δ minimal for this property. We set $G := \mathbf{G}^F$, the finite group of fixed points.

We fix a pair (\mathbf{T}, \mathbf{B}) consisting of a maximal torus contained in a Borel subgroup of \mathbf{G} , both of which are assumed to be F -stable. We denote by W the Weyl group of \mathbf{G} , and by S the set of simple reflections in W associated with \mathbf{B} . Then F^δ acts trivially on W . Following Lusztig [15, 17.2], for any F -stable irreducible character χ of W , one can choose a preferred extension $\tilde{\chi}$ of χ to the group $W \rtimes \langle F \rangle$ satisfying $\tilde{\chi}(xF^\delta) = \tilde{\chi}(x)$ for all $x \in W \rtimes \langle F \rangle$. The almost character associated to $\tilde{\chi}$ is then the following uniform character

$$R_{\tilde{\chi}} = \frac{1}{|W|} \sum_{w \in W} \tilde{\chi}(wF) R_{\mathbf{T}_{wF}}^{\mathbf{G}}(1).$$

Let $X(w)$ denote the Deligne–Lusztig variety associated with w . Then its ℓ -adic cohomology groups $H^i(X(w))$ with coefficients in $K \supset \mathbb{Q}_\ell$ give rise to the so-called *unipotent* representations of G . By [14, Thm. 4.23] the decomposition of almost characters in terms of unipotent characters can be computed explicitly. Conversely, for $w \in W$ the orthogonality relations for the Deligne–Lusztig characters yield

$$R_{\mathbf{T}_{wF}}^{\mathbf{G}}(1) = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(X(w))] = \sum_{\chi \in (\text{Irr } W)^F} \tilde{\chi}(wF) R_{\tilde{\chi}}$$

as a virtual KG -module.

The Frobenius F^δ acts on the Deligne–Lusztig variety $X(w)$, making the cohomology groups $H^i(X(w))$ into $G \times \langle F^\delta \rangle$ -modules. Digne and Michel [1] have extended the previous formula to take into account this action. By [13, Cor. 3.9], the eigenvalues of F^δ on a unipotent character ρ in the cohomology of $X(w)$ are of the form $\lambda_\rho q^{\delta m/2}$ where λ_ρ is a root of unity which depends only on ρ and m is a nonnegative integer. We fix an indeterminate v and we shall denote by $v^{\delta m} \rho$ the class in the Grothendieck group of $G \times \langle F^\delta \rangle$ -modules of such a representation.

Let $\mathcal{H}_v(W)$ be the Iwahori–Hecke algebra of W with equal parameters v . By convention, the standard basis $(t_w)_{w \in W}$ of $\mathcal{H}_v(W)$ will satisfy the relation $(t_s + v)(t_s - v^{-1}) = 0$ for all $s \in S$. For $\chi \in (\text{Irr } W)^F$ we denote by $\tilde{\chi}_v$ the character of $\mathcal{H}_v(W) \rtimes \langle F \rangle$ which specializes to $\tilde{\chi}$ at $v = 1$. We denote by $(C'_w)_{w \in W}$ (resp. $(C_w)_{w \in W}$) the Kazhdan–Lusztig basis (resp. twisted Kazhdan–Lusztig basis) of $\mathcal{H}_v(W)$. For any simple reflection $s \in S$ we have $C'_s = t_s + v$ and $C_s = t_s - v^{-1}$. If ι denotes the $\mathbb{Z}[v, v^{-1}]$ -linear involution on $\mathcal{H}_v(W)$ defined by $\iota(t_w) = (-1)^{\ell(w)} t_{w^{-1}}^{-1}$ then $\iota(C'_w) = (-1)^{\ell(w)} C_w$. The virtual $G \times \langle F^\delta \rangle$ -modules afforded by the intersection cohomology of Deligne–Lusztig varieties can be computed by means of the C'_w base. For the following result, see [14, Thm. 3.8]:

Theorem 2.1 (Lusztig). *Let $w \in W$. The class in the Grothendieck group of $G \times \langle F^\delta \rangle$ -modules of the intersection cohomology of $X(w)$ is given by*

$$\sum_{i \in \mathbb{Z}} (-1)^i [IH^i(\overline{X(w)})] = (-1)^{\ell(w)} \sum_{\chi \in (\text{Irr } W)^F} \tilde{\chi}_v(C'_w F) R_{\tilde{\chi}}.$$

Example 2.2. The group $G = \text{SL}_2(q)$ has two unipotent characters: the trivial character 1_G and the Steinberg character St_G . Here, the element t_s represents the cohomology of $X(s)$ whereas the element $C'_s = t_s + v$ represents the cohomology of $\overline{X(s)} \cong \mathbf{P}_1$. The irreducible characters of $\mathcal{H}_v(W)$ corresponding to the trivial and sign characters of W are defined by $1_v(t_s) = v^{-1}$ and $\text{sgn}_v(t_s) = -v$. We have $[H^0(X(s))] = 1_G$ and $[H^1(X(s))] = v^2 \text{St}_G$ so that

$$[H^0(X(s))] - [H^1(X(s))] = 1_G - v^2 \text{St}_G.$$

It corresponds to the element vt_s since $1_v(vt_s) = 1$ and $\text{sgn}_v(vt_s) = -v^2$.

We have also $[H^0(\overline{X(s)})] = 1_G$, $[H^1(\overline{X(s)})] = 0$ and $[H^2(\overline{X(s)})] = v^2 1_G$ so that

$$[H^0(\overline{X(s)})] - [H^1(\overline{X(s)})] + [H^2(\overline{X(s)})] = 1_G + v^2 1_G$$

corresponds to vC'_s since $1_v(vC'_s) = v(v^{-1} + v) = 1 + v^2$ and $\text{sgn}_v(vC'_s) = v(-v + v) = 0$. Since $\overline{X(s)}$ is smooth and one-dimensional, $IH^i(\overline{X(s)}) = H^{i+1}(\overline{X(s)})(-1)$, where (-1) is a Tate twist (contributing v^{-1} in the character). Consequently the intersection cohomology of $\overline{X(s)}$ corresponds to $-C'_s$.

2.2. Basic sets for finite reductive groups. Let ℓ be a prime number and (K, \mathcal{O}, k) be an ℓ -modular system. We assume that it is large enough for all the finite groups encountered. Furthermore, since we will be working with ℓ -adic cohomology we will assume throughout this note that K is a finite extension of \mathbb{Q}_ℓ .

Let H be a finite group. Representations of H will always be assumed to be finite-dimensional. Recall that every projective kH -module lifts to a representation of H over K . The character afforded by such a representation will be referred to as a *projective character*. Integral linear combinations of projective characters will be called *virtual projective characters*.

Throughout this paper, we shall make the following assumptions on ℓ :

- $\ell \neq p$ (non-defining characteristic),
- ℓ is good for \mathbf{G} and $\ell \nmid |(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$.

In this situation, the unipotent characters lying in a given unipotent ℓ -block of G form a *basic set* of this block [7, 6]. Consequently, the restriction of the decomposition matrix of the block to the unipotent characters is invertible. In particular every (virtual) unipotent character is a virtual projective character, up to adding and removing some non-unipotent characters.

2.3. A positivity conjecture . Let D_G denote the Alvis–Curtis duality, with the convention that if ρ is a cuspidal unipotent character, then $D_G(\rho) = (-1)^{\text{rk}_F(\mathbf{G})} \rho$ where $\text{rk}_F(\mathbf{G}) = |S/F|$ is the F -semisimple rank of \mathbf{G} . Then $D_G(R_{\tilde{\chi}}) = R_{\tilde{\chi} \otimes \widetilde{\text{sgn}}}$ (see [14, 6.8.6])

and $(\tilde{\chi} \otimes \widetilde{\text{sgn}})_v = \tilde{\chi}_v \circ \iota$ (see for example [14, 5.11.4]) so that by Theorem 2.1

$$\sum_{i \in \mathbb{Z}} (-1)^i [D_G(IH^i(\overline{X(w)}))] = \sum_{\chi \in (\text{Irr } W)^F} \tilde{\chi}_v(C_w F) R_{\tilde{\chi}}.$$

We denote by Q_w the restriction to G of this virtual character, and by $Q_w[\lambda]$ the generalized λ -eigenspace of F^δ for $\lambda \in k^\times$.

Lusztig proved [14, Prop. 6.9 and 6.10] that up to a global sign, Q_w (and even $Q_w[\lambda]$) is always a nonnegative combination of unipotent characters (note that the assumption on q can be removed by [4, Cor. 3.3.22]). The sign is given by the a -value $a(w)$ of the two-sided cell in which w lies.

Proposition 2.3 (Lusztig). *For all $w \in W$ and all $\lambda \in k^\times$, $(-1)^{a(w)} Q_w[\lambda]$ is a sum of unipotent characters.*

Unipotent characters are only the unipotent part of virtual projective characters in general. We conjecture that the modular analogue of Proposition 2.3 should hold in general, that is that Q_w is actually a proper projective character whenever ℓ is not too small.

Conjecture 2.4. *Under the assumption on ℓ in §2.2, for all $w \in W$ and all $\lambda \in k^\times$, $(-1)^{a(w)} Q_w[\lambda]$ is the unipotent part of a projective character.*

Example 2.5. The closure of the Deligne–Lusztig variety $X(w_0)$ associated with the longest element w_0 of W is smooth and equal to \mathbf{G}/\mathbf{B} . Therefore its intersection cohomology $IH^\bullet(\overline{X(w_0)})$ consists of copies of the trivial representation in degrees of a given parity. Since $a(w_0) = \ell(w_0)$, then $(-1)^{a(w_0)} Q_{w_0}$ is a nonnegative multiple of the Steinberg character. On the other hand, the Steinberg character is the unipotent part of a unique projective indecomposable module (given by a summand of a Gelfand–Graev module) and therefore the conjecture holds for w_0 .

Remark 2.6. This conjecture has been checked on many decomposition matrices, including the unipotent blocks with cyclic defect groups for exceptional groups, and the matrices that are determined in [5, 2]. We will give some examples in § 4.

3. APPLICATIONS

In this section we explain how to deduce properties of decomposition matrices using Conjecture 2.4.

3.1. Families of simple unipotent modules. Following [14, §5], we denote by \leq_{LR} the partial order on two-sided cells of W . Recall that to each cell Γ corresponds a two-sided ideal $I_{\leq \Gamma} := \text{span}_{\mathbb{Q}}\{C_{w|v=1} \mid w \leq_{\text{LR}} \Gamma\}$ of the group algebra $\mathbb{Q}W$. Moreover, given $\chi \in \text{Irr } W$, there is a unique two-sided cell Γ_χ such that χ occurs in $I_{\leq \Gamma_\chi}/I_{< \Gamma_\chi}$. To each two-sided cell Γ one can attach a so-called family \mathcal{F}_Γ of unipotent characters. They are defined as the constituents of R_χ for various χ such that $\Gamma_\chi = \Gamma$. By [14, Thm. 6.17], they form a partition of the set of unipotent characters of G .

We can use the characters Q_w to define families of simple unipotent kG -modules. For each Q_w we choose a virtual projective kG -module \tilde{Q}_w whose character coincides with Q_w up to adding/removing non-unipotent characters. We denote by $\text{Irr}_{\leq \Gamma} kG$ (resp. $\text{Irr}_{< \Gamma} kG$)

the set of simple unipotent kG -modules N such that a projective cover P_N occurs in the virtual module \tilde{Q}_w for some $w \leq_{\text{LR}} \Gamma$ (resp. $w <_{\text{LR}} \Gamma$). We define the family of simple unipotent modules associated to Γ as $\text{Irr}_{\Gamma} kG = \text{Irr}_{\leq \Gamma} kG \setminus \text{Irr}_{< \Gamma} kG$. Since the regular representation is uniform (see for example [3, Cor. 12.14]) then every indecomposable projective module lying in a unipotent block occurs in some \tilde{Q}_w , therefore every simple unipotent module belongs to at least one family. However it is unclear whether this family is unique in general.

Proposition 3.1. *Let b be a unipotent ℓ -block of G . Assume that*

- (i) *every unipotent character in b is a linear combination of bR_{χ} 's, and*
- (ii) *Conjecture 2.4 holds.*

Then for every two-sided cell Γ

$$|\text{Irr}_{\Gamma} b kG| = |\mathcal{F}_{\Gamma} \cap \text{Irr } b|.$$

In particular, any simple b -module lies in a unique family.

Proof. Given $\chi \in \text{Irr } W$ and Γ_{χ} the corresponding two-sided cell, the primitive idempotent $e_{\chi} \in \mathbb{Q}W$ lies in $I_{\leq \Gamma_{\chi}}$. In other words, it is a linear combination of $C_{w|v=1}$'s with $w \leq_{\text{LR}} \Gamma_{\chi}$. Consequently, for all $\chi \in \text{Irr } W$

$$R_{\chi} \in \text{span}\{Q_w \mid w \leq_{\text{LR}} \Gamma_{\chi}\}.$$

Therefore $|\text{Irr}_{\leq \Gamma} b kG| \geq \dim \text{span}\{bR_{\chi} \mid \Gamma_{\chi} \leq \Gamma\}$ which is the number of unipotent characters in b whose family is smaller than Γ . But every bQ_w is a linear combination of these characters, and so is every PIM occurring in $b\tilde{Q}_w$ if Conjecture 2.4 holds. In particular, $|\text{Irr}_{\leq \Gamma} b kG|$ has to be smaller than the number of these characters and we conclude using the assumption (i). \square

Remark 3.2. Note that (i) is not always satisfied, for example when two complex conjugate characters lie in the same block. However the validity of this assumption is easy to check on the Fourier matrices. For example, it is valid whenever $\ell|(q \pm 1)$ and \mathbf{G} is exceptional.

3.2. Application to the decomposition matrix. It is conjectured that, for ℓ not too small, the ℓ -modular decomposition matrix of G depends only on the order of q modulo ℓ . The following proposition gives some evidence toward this conjecture as well as to Geck's conjecture on the unitriangular shape of the decomposition matrix [8, Conj. 3.4].

Proposition 3.3. *Let b be a unipotent ℓ -block of G . Assume that*

- (i) *Conjecture 2.4 holds,*
- (ii) *$|\text{Irr}_{\Gamma} b kG| = |\mathcal{F}_{\Gamma} \cap \text{Irr } b|$ for any two-sided cell Γ of W .*

Then the unipotent part of the decomposition matrix of b has the following shape:

$$D_{\text{uni}} = \begin{pmatrix} D_{\mathcal{F}_1} & 0 & \cdots & 0 \\ * & D_{\mathcal{F}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & D_{\mathcal{F}_r} \end{pmatrix}$$

where \mathcal{F}_i runs over the families and where each $D_{\mathcal{F}_i}$ is a square matrix of size $|\mathcal{F}_i \cap \text{Irr } b|$. Furthermore, the entries of D_{uni} are bounded above independently of q and ℓ .

Proof. Given any simple kG -module N in the block, there exists by (ii) a unique two-sided cell Γ such that $N \in \text{Irr}_\Gamma bkG$ and P_N occurs in some Q_w for $w \in \Gamma$ (note that $w \not\prec_{\text{LR}} \Gamma$ otherwise N would belong to a smaller family). Since the matrix of the Q_w 's is block triangular with respect to families, the proposition follows from (i). \square

3.3. Determining decomposition numbers. The bounds on the entries of the decomposition matrix given by the Q_w 's are often small enough to determine some of the decomposition numbers. We illustrate this on the principal Φ_2 -block of the group $F_4(q)$.

Proposition 3.4. *Assume that $(q, 6) = 1$, $q \equiv -1 \pmod{\ell}$ and $\ell > 11$. If Conjecture 2.4 holds, then the following matrix*

$$\begin{array}{c|ccccc} \phi_{9,10} & 1 & & & & \\ \phi''_{2,16} & 1 & 1 & & & \\ \phi'_{2,16} & 1 & . & 1 & & \\ B_{2,\varepsilon} & 2 & . & . & 1 & \\ \phi_{1,24} & 3 & 2 & 2 & 4 & 1 \end{array}$$

is a submatrix of the ℓ -modular decomposition matrix of $F_4(q)$. (Here, the "."s denote zero entries.)

Proof. By [12], there exist integers f, g, h, i, j with $f, h, i \geq 2$, $g \geq 4f - 5$ and $j \geq 4$ such that the matrix

$$\begin{array}{c|ccccc} \phi_{9,10} & 1 & & & & \\ \phi''_{2,16} & 1 & 1 & & & \\ \phi'_{2,16} & 1 & . & 1 & & \\ B_{2,\varepsilon} & f & . & . & 1 & \\ \phi_{1,24} & g & h & i & j & 1 \end{array}$$

is a submatrix of the decomposition matrix of $F_4(q)$. Let b be the block idempotent associated with the principal ℓ -block of $F_4(q)$. With $w = s_2 s_3 s_2 s_3 s_4 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1$ we have

$$bQ_w = 32\phi_{9,10} + 72\phi''_{2,16} + 72\phi'_{2,16} + 72B_{2,\varepsilon} + 288\phi_{1,24}.$$

If we denote by P_1, \dots, P_5 the unipotent part of the characters of the PIMs corresponding to the last five columns of the decomposition matrix of $F_4(q)$, we can decompose Q_w as

$$bQ_w = 32P_1 + 40P_2 + 40P_3 + (72 - 32f)P_4 + (288 - 32g - 40h - 40i - (70 - 32f)j)P_5.$$

By Conjecture 2.4 the multiplicity of each P_k should be nonnegative, that is $72 - 32f \geq 0$ and $288 - 32g - 40h - 40i - (70 - 32f)j \geq 0$. Since $f \geq 2$ the first relation forces $f = 2$ (and therefore $g \geq 3$). The second becomes $288 - 32g - 40h - 40i - 8j \geq 0$. Since $288 = 32 + 40 \times 2 + 40 \times 2 + 8 \times 4$ is the minimal value that the expression $32g + 40h + 40i + 8j$ can take, we deduce that $h = i = 2$, $g = 3$ and $j = 4$. \square

4. SOME EVIDENCE

4.1. A cuspidal module in the unitary group. We give here a non-trivial example for $\text{SU}_n(q)$ where Conjecture 2.4 holds. The key point is to find a formula for $(C_w F)|_{v=1}$ in terms of well-identified elements of W . This is done using the geometric description of Kazhdan–Lusztig polynomials. The proof given here can be adapted to other groups, even when the Schubert variety is no longer smooth, using Bott–Samelson varieties instead.

Recall that the set of unipotent characters of $\mathrm{SU}_n(q)$ is parametrized by partitions of n . Given such a partition λ , we denote by ρ_λ (resp. χ_λ) the corresponding unipotent character (resp. character of \mathfrak{S}_n), with the convention that ρ_{1^n} is the Steinberg character.

Proposition 4.1. *Let $\ell > n$ be a prime dividing $q + 1$. Then Conjecture 2.4 holds for $\mathrm{SU}_n(q)$ and $w = s_1 w_0 \in \mathfrak{S}_n$. Furthermore,*

$$\frac{(-1)^{a(w)}}{(n-1)!} Q_w = \rho_{21^{n-2}} + (n-1)\rho_{1^n}$$

is the unipotent part of a projective indecomposable $k\mathrm{SU}_n(q)$ -module.

Here w_0 is the longest element of $W \cong \mathfrak{S}_n$ and s_i is the transposition $(i, i+1)$. For computing Q_w for $w = s_1 w_0$ we first need to compute the decomposition of the corresponding Kazhdan–Lusztig element on the standard basis:

Lemma 4.2. *If w_0 is the longest element of W and w_I the longest element of W_I with $I = \{s_1, \dots, s_{n-2}\}$ we have*

$$C'_{s_1 w_0} = v^{-1} C'_{w_0} - v^{n-2} t_{s_1 s_2 \dots s_{n-1}} C'_{w_I}.$$

Proof of the lemma. Let \mathbf{B} be a Borel subgroup of $\mathbf{G} = \mathrm{SL}_n$ and \mathbf{P} be the standard parabolic subgroup corresponding to the set of simple reflections $I = \{s_1, s_2, \dots, s_{n-2}\}$. Let $\pi : \mathbf{G}/\mathbf{B} \rightarrow \mathbf{G}/\mathbf{P}$ be the canonical projection. The variety \mathbf{G}/\mathbf{P} is isomorphic to the projective space \mathbf{P}_{n-1} and is paved by the affine spaces $\mathbf{B}s_i s_{i+1} \dots s_{n-1} \mathbf{P}/\mathbf{P}$ of dimension $n - i$. The closure of each of them is in turn a projective space of dimension $n - i$, and hence is smooth.

Each element $w \in W$ can be written uniquely as $w = s_i s_{i+1} \dots s_{n-1} x$ with $1 \leq i \leq n$ and $x \in W_I$. In particular, the image of the corresponding Schubert cell under π is exactly $\mathbf{B}s_i s_{i+1} \dots s_{n-1} \mathbf{P}/\mathbf{P}$. We deduce that $\overline{\mathbf{B}s_1 w_0 \mathbf{B}/\mathbf{B}} = \pi^{-1}(\overline{\mathbf{B}s_2 s_3 \dots s_{n-1} \mathbf{P}/\mathbf{P}})$. In particular, it is smooth and by [11, Thm. A2] the Kazhdan–Lusztig element is given by

$$C'_{s_1 w_0} = \sum_{w \leq s_1 w_0} v^{\ell(s_1 w_0) - \ell(w)} t_w = v^{-1} \left(\sum_{w \in W} v^{\ell(w_0) - \ell(w)} t_w - \sum_{w \not\leq s_1 w_0} v^{\ell(w_0) - \ell(w)} t_w \right).$$

Now $w \not\leq s_1 w_0$ if and only if $w = s_1 s_2 \dots s_{n-1} x$ with $x \in W_I$. In that case one can write $t_w = t_{s_1 \dots s_{n-1}} t_x$ and the result follows from the relation $\ell(w) - \ell(w_I) = n - 1$ and the expression for C'_v when v is the longest element of a parabolic subgroup (which again comes from the smoothness of $\overline{\mathbf{B}w_I \mathbf{B}/\mathbf{B}} = \mathbf{P}_I/\mathbf{B}$). \square

Proof of Proposition 4.1. By definition,

$$Q_w = \sum_{i \in \mathbb{Z}} (-1)^i [D_G(IH^i(\overline{X(w)}))]_{|v=1} = \sum_{\chi \in (\mathrm{Irr} W)^F} \tilde{\chi}_v(C_w F)_{|v=1} R_{\tilde{\chi}}.$$

Now using the involution $\iota : t_w \mapsto (-1)^{\ell(w)} t_{w^{-1}}$ of $\mathcal{H}_v(W)$ one has $\iota(C'_w) = (-1)^{\ell(w)} C_w$, so that from Lemma 4.2 we get $C_{s_1 w_0} = -v^{-1} C_{w_0} + v^{n-2} t_{s_{n-1} \dots s_1}^{-1} C_{w_I}$. The evaluation at $v = 1$ yields

$$\begin{aligned} (C_{s_1 w_0})_{|v=1} &= - \sum_{w \in W} (-1)^{\ell(w_0) - \ell(w)} w + s_1 \dots s_{n-1} \sum_{w \in W_I} (-1)^{\ell(w_I) - \ell(w)} w \\ &= (-1)^{\ell(w_0)+1} |W| e_{\mathrm{sgn}} + (-1)^{\ell(w_I)} |W_I| s_1 \dots s_{n-1} e_{\mathrm{sgn}_I} \end{aligned}$$

where sgn (resp. sgn_I) is the sign character of W (resp. W_I) and e_χ denotes the central idempotent corresponding to the character χ . Recall from [15, 17.2], that the extension $\tilde{\chi}$ of χ satisfies $\tilde{\chi}(wF) = (-1)^{a(\chi)}\chi(w w_0)$. Using the fact that $w_0 = s_{n-1} \cdots s_1 w_{F(I)}$ we obtain

$$\begin{aligned} (-1)^{a(\chi)} \tilde{\chi}_v(C_{s_1 w_0} F)|_{v=1} &= (-1)^{\ell(w_0)+1} |W| \chi(e_{\text{sgn}} w_0) + (-1)^{\ell(w_I)} |W_I| \chi(w_{F(I)} e_{\text{sgn}_{F(I)}}) \\ &= -|W| \chi(e_{\text{sgn}}) + |W_I| \chi(e_{\text{sgn}_{F(I)}}). \end{aligned}$$

Since $\chi(e_{\text{sgn}_{F(I)}}) = (\text{Res}_{W_{F(I)}}^W \chi)(e_{\text{sgn}_{F(I)}})$, we deduce that it is zero whenever χ is not a constituent of $\text{Ind}_{W_{F(I)}}^W \text{sgn}_{F(I)}$. The two constituents of this induced representation correspond to the partitions 1^n and 21^{n-2} , with respective A -functions given by $\ell(w_0)$ and $\ell(w_0) - 1$. Now using the fact that $R_{\tilde{\chi}_\mu} = (-1)^{a(\mu)+A(\mu)} \rho_\mu$ for any partition $\mu \vdash n$, one finds

$$Q_{s_1 w_0} = (-1)^{\ell(w_0)-1} |W_I| (\rho_{21^{n-2}} + (n-1) \rho_{1^n}),$$

and $\rho_{21^{n-2}} + (n-1) \rho_{1^n}$ is the unipotent part of the character of an indecomposable projective module by [2, Thm. 5.9]. \square

4.2. Groups of small rank. We finish by computing for several groups G of small rank the contribution to the principal ℓ -block b by various Q_w 's. In the table, we give in the second column the minimal integer d such that $\ell \mid (q^d - 1)$, and in the last column the decomposition of Q_w in the basis of projective indecomposable modules which is obtained from the decomposition matrices in [10, 5, 2, 9]. Note however that this might change when ℓ is small as the decomposition matrix will change (see for example $G_2(4)$ with $\ell = 5$). In any case, Q_w remains the unipotent part of a proper projective character. Our notation for the unipotent characters is as in the cited sources.

G	d	w	λ	$bQ_w[\lambda]$
$\text{Sp}_6(q)$	2	$s_1 s_2 s_1 s_2 s_3$	1	$3(\rho_{1,1^2} + \text{St}) + (B_2, 1^2 + 2\text{St}) + (\rho_{1^3, -} + 3\text{St})$
$G_2(q)$	2	$s_1 s_2$	-1	$G_2[-1] + 2\text{St}$
${}^3\text{D}_4(q)$	6	$s_1 s_2 s_3 s_2$	1	$2(\phi_{2,2} + \phi'_{1,3} + \text{St}) + 2({}^3\text{D}_4[-1] + 2\text{St})$
$\text{SU}_6(q)$	2	$s_1 s_3 s_4 s_3$	1	$2(\rho_{321} + 2\rho_{2^3} + 2\rho_{31^3} + 2\rho_{2^2 1^2} + 2\rho_{21^4} + 6\rho_{1^6})$
$F_4(q)$	12	$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_3$	ζ_{12}^5 ζ_{12}^4	$2F_4[\theta] + 20B_2, \text{St} + 4\text{St}$ $4F_4[\theta] + F_4[\text{i}] + 11B_2, \text{St} + 10\text{St}$

One can even construct larger examples for which the conjecture holds. We give below five examples of computation of Q_w in $\text{SU}_{10}(q)$ when $q \equiv -1 \pmod{\ell}$. When $\ell > 17$, we can use [2, Thm. 6.2] to decompose them on the basis of projective indecomposable modules. It turns out that at least four of them are (up to a scalar) the character of a

projective indecomposable module.

w	Q_w
$s_1 s_3 s_4 s_3 s_5 s_4 s_3 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_3 s_9 s_8 s_7 s_6 s_5 s_4 s_3$	$1440\Psi_{321^5}$
$s_1 s_2 s_1 s_4 s_5 s_4 s_6 s_5 s_4 s_7 s_6 s_5 s_4 s_8 s_7 s_6 s_5 s_4$	$96\Psi_{32^2 1^3}$
$s_5 s_4 s_3 s_2 s_1 s_4 s_3 s_2 s_4 s_3 s_4 s_9 s_8 s_7 s_9 s_8 s_9$	$576\Psi_{32^3 1}$
$s_1 s_2 s_1 s_4 s_5 s_4 s_6 s_5 s_4 s_7 s_6 s_5 s_4 s_9$	$96\Psi_{3^2 2 1^2}$
$s_2 s_1 s_2 s_3 s_7 s_6 s_5 s_7 s_6 s_7 s_9$	$16(\Psi_{4321} + (1 - \alpha)\Psi_{32^3 1} + \beta\Psi_{2^4 1^2})$

Here $\alpha, \beta \in \{0, 1\}$ are as in [2, Thm. 6.2]

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